Horn of singularities for the Stark-Wannier ladders

1991 J. Phys. A: Math. Gen. 24 4275

(http://iopscience.iop.org/0305-4470/24/18/015)

J. Phys. A: Math. Gen. 24 (1991) 4275–4282. Printed in the UK

Horn of singularities for the Stark–Wannier ladders*

Vincenzo Grecchi†, Marco Maioli‡‖ and Andrea Sacchetti§

† Dipartimento di Matematica, Università degli Studi di Bologna, I-40127 Bologna, Italy
‡ Dipartimento di Matematica, Università degli Studi della Basilicata, I-85100 Potenza, Italy
§ Dipartimento di Matematica Pura ed Applicata, Università degli Studi di Modena, I-41100 Modena, Italy

Received 3 January 1991, in final form 15 May 1991

Abstract. We prove that the small field asymptotic behaviour of the Stark–Wannier ladders near the real direction is generically highly singular. This result is in agreement with the conjecture of a chaotic behaviour of the lifetime of the states because of infinitely many crossings.

Since the original construction of the Stark–Wannier ladders in the single-band approximation (Wannier 1960) the problem of the existence and main features of their states was posed. It was soon pointed out that they are asymptotic, up to any order, to the formal perturbative series (Nenciu and Nenciu 1981) and that they should be sharp resonances (or bound states in singular cases as suggested by Berezhkovskii and Ovchinnikov 1976 and Bentosela *et al* 1982a). It was also evident that the main technical problem comes from the asymptotic density of levels for small field and from the many possible crossings.

The analysis (both mathematical and numerical) of such crossings on suitable models showed a chaotic behaviour of the widths, associated with the (avoided) crossing phenomenon for the levels (see Avron 1982 and Bentosela *et al* 1982b). The numerical and analytical study extended to the complex electric field (Ferrari *et al* 1985) gave a more direct analysis of the crossing effects as cuts of Bender–Wu type (see Bender and Wu 1969, Crutchfield 1978).

Meanwhile the rigorous study in complex field started by Avron (1979) led to the discovery of the asymptotics in any complex direction (Bentosela *et al* 1988). On the other hand, the asymptotics in the real direction was established by Nenciu and Nenciu (1981) to all orders for each pseudoeigenvalue, or resonance (if existent in this region of the parameters).

In this work we make precise the asymptotics result for each resonance (as defined by Herbst and Howland 1981) at a small real field, in order to compare it with the complex direction asymptotics already calculated. The most interesting result (see proposition 1) is that such an expansion is generically different, up to second order (the next one beyond the Wannier approximation), from the one in the complex direction. In particular, for the first ladder the second-order coefficient in the asymptotic expansion in the complex direction vanishes, while the real-direction one diverges, when the strength of the periodic potential tends to zero.

* Partially supported by Ministero della Università e della Ricerca Scientifica e Tecnologica.
‖ Partially supported by Dipartimento di Matematica Pura ed Applicata, Università di Modena, I-41100 Modena, Italy.

0305-4470/91/184275+08$03.50

Bounded analyticity is excluded by such different expansions in any finite sector containing the positive ray as made precise in proposition 2 and related remarks. Note that the existence of a natural boundary line tangent to the real axis at the origin (as in the model proposed by Bessis *et al* (1989) in the context of symplectic maps or a sequence of divergence, in addition to or in place of a sequence of crossing-type singularities, is not excluded. In any case the width of each resonance is certainly sensitive to such analyticity structures as can be seen in the finite crystal model (Bentosela *et al* 1982b, Ferrari *et al* 1985).

Now we discuss the situation for complex values of the electric field parameter following the treatment of Bentosela *et al* 1988. Let us consider, as above, a translation analytic even potential $V(x)=V(x+a)$, $a>0$, of the Herbst–Howland type with the further condition that all the gaps of the spectrum are open. In particular, we consider the translated Hamiltonian operator, in order to define the resonances in the limit $\eta\to 0$:

$$H_\alpha(F+\mathrm{i}\eta):=p^2+V(x+\alpha)+(F+\mathrm{i}\eta)(x+\alpha) \qquad \alpha\in\mathbb{C}. \tag{1}$$

It is easy to show that $H_\alpha(F+\mathrm{i}\eta)$ is an analytic family of operators for $\eta>0$ and $0<\Im\alpha<b$, for some $b>0$. The spectrum, which is discrete and consists of ladders of eigenvalues

$$\{\lambda_{n,j}(F+\mathrm{i}\eta)\}_{n\in\mathbb{N},j\in\mathbb{Z}}=\{\lambda_{n,0}(F+\mathrm{i}\eta)+ja(F+\mathrm{i}\eta)\}_{n\in\mathbb{N},j\in\mathbb{Z}} \qquad \eta>0 \tag{2}$$

is independent of α and such eigenvalues are analytic at least in the disks $B_{R_n}(\mathrm{i}R_n)$. Besides, for $\eta=0$ and $\Im\alpha=b>0$, the operator $H_\alpha(F)$ defines the resonances lying in the strip $S_b:=\{z\in\mathbb{C}\mid -bF<\Im z\leqslant 0\}$. Following the latest results given by Buslaev and Dmitrieva (1987, 1989), Bentosela and Grecchi (1990) and Combes and Hislop (1990), we suppose the existence of ladders of resonances $\lambda_{n,j}(F)$ having the Wannier states as a small field limit. One can prove that a strong resolvent limit exists for the family $H_\alpha(F+\mathrm{i}\eta)$ as $\eta\to 0$, for α and $F\neq 0$ fixed. Thus the resonances are, in general, the limits of the eigenvalues of $H_\alpha(F+\mathrm{i}\eta)$ as $\eta\to 0$. As a minimal hypothesis let an analytic continuation of the eigenvalues $\lambda_{n,j}$ exist from the disks $B_{R_n}(\mathrm{i}R_n)$ to a strip $-R_n\leqslant\Re z\leqslant R_n$, $\Im z>0$ and, defining the continuation in a suitable way, such that

$$\lim_{\eta\to 0}\lambda_{n,j}(F+\mathrm{i}\eta)=\lambda_{n,j}(F) \qquad F>0 \tag{3}$$

if they lie in the strip. Since the family $H_\alpha(F+\mathrm{i}\eta)$ is analytic, the singularities of the eigenvalues $\lambda_{n,j}(F+\mathrm{i}\eta)$ are given by the possible crossings, that is they consist of algebraic branch points, up to accumulation effects.

Here is given the main result concerning the difference between the asymptotic expansion to the second order for real and complex electric field. Such a result is obtained, as mentioned previously under the following assumption.

Hypothesis. There exists a ladder of resonances $\lambda_{n,j}(F)$ for small $F>0$, such that $\lambda_{n,0}(F)\to\langle E_n\rangle$ as $F\to 0$, where $\langle E_n\rangle:=(a/2\pi)\int_{\mathscr{B}}E_n(k)\,\mathrm{d}k$. Here $\mathscr{B}$ is the Brillouin zone, i.e. the torus $\mathbb{R}/(2\pi/a)$.

In this case, the Nenciu–Nenciu analysis guarantees the asymptotic expansion to all orders of $\lambda_{n,j}(F)$ as pseudoeigenvalues.

Proposition 1. *Let* c_n be the second-order coefficient of the asymptotic expansion of the eigenvalue $\lambda_{n,j}(F+\mathrm{i}\eta)$ of $H_\alpha(F+\mathrm{i}\eta)$, analytic in the disk $B_{R_n}(\mathrm{i}R_n)$, so that

$$\lambda_{n,j}(F+\mathrm{i}\eta)=\langle E_n\rangle+ja(F+\mathrm{i}\eta)+c_n(F+\mathrm{i}\eta)^2+O((F+\mathrm{i}\eta)^{5/2})$$
$$\text{as}\quad |F+\mathrm{i}\eta|\to 0 \tag{4}$$

where $F+i\eta$ lies in any sector $|\pi/2-\arg(F+i\eta)|\leq\theta, 0<\theta<\pi/2$, where $n\in\mathbb{N}, j\in\mathbb{Z}$. Let d_n be the second-order coefficient of the asymptotic expansion of the resonances $\lambda_{n,j}(F)$ of $H_\alpha(F)$, as $F\to 0^+$, so that

$$\lambda_{n,j}(F)=\langle E_n\rangle+jaF+d_nF^2+O(F^3). \tag{5}$$

Then, generically, $c_n\neq d_n$.

Proof. We compute the coefficients of the asymptotic series using the crystal momentum representation (CMR) (see Bentosela *et al* 1988) in the real case (for the complex direction of the electric field see Bentosela *et al* (1988)). In such a representation the operator $H_\alpha(0)$ becomes the α-independent diagonal matrix $\tilde{H}_0(0)=\boldsymbol{E}$ since the Wannier states $w_n^k=\{w_n^k(K)\}_{K\in\mathbb{Z}}$ by analytic translation become $w_{n,\alpha}^k=\mathrm{e}^{ik\alpha}\,(\mathrm{e}^{i\boldsymbol{K}\alpha}w_n^k)$, where $\boldsymbol{K}$ is defined by $(\boldsymbol{K}w_n^k)\,(K):=Kw_n^k(K)$, and they are in $l^2(\mathbb{Z})$ for $0<|\Im\alpha|<b$. More generally the operator $H_\alpha(F+i\eta)$ becomes

$$\tilde{H}_\alpha(F+i\eta)=\tilde{H}_0(F+i\eta)+\alpha(F+i\eta)=\mathbf{E}+(F+i\eta)[\mathbf{X}+i\mathbf{D}]+\alpha(F+i\eta) \tag{6}$$

where $(\boldsymbol{X})_{n,m}(k)\equiv X_{n,m}(k):=i\langle w_n^k, \partial w_m^k/\partial k\rangle_{l^2}$ for $0<|\Im\alpha|<b$ and $\boldsymbol{D}$ is the derivative operator $(\boldsymbol{D}a)_{n,m}:=\delta_n^m(\partial a_n/\partial k)$.

Now let $n\in\mathbb{N}$ be fixed. In Bentosela *et al* (1988) it is proved that (4) holds with $c_m=f_n(\bar{k}_n)$, where $f_n(k)$ is the even analytic function defined on the Brillouin zone $\mathscr{B}$ by †

$$f_n(k):=\sum_{m\neq n}\frac{|X_{n,m}(k)|^2}{E_n(k)-E_m(k)} \tag{7}$$

and $\bar{k}_n$ is uniquely determined by $E_n(\bar{k}_n)=\langle E_n\rangle$, $0<\bar{k}_n<\pi/a$. As we shall prove we have $c_n-d_n=\tilde{f}_n(\bar{k}_n)$, where

$$\tilde{\varphi}(k):=\varphi(k)-\langle\varphi\rangle=\varphi(k)-\frac{a}{2\pi}\int_{\mathscr{B}}\varphi(k)\,\mathrm{d}k. \tag{8}$$

Since $\tilde{f}_n(k)$ is analytic and not identically zero in $(0, \pi/a)$, it has a finite number of zeros in any compact contained in $(0, \pi/a)$, so that $\tilde{f}_n(\bar{k}_n)=c_n-d_n\neq 0$ generically. In fact there is no particular relation between $\tilde{f}_n$ and E_n, as can be seen from the definition (formula (7)). In any case, the proof of non-existence of a hidden symmetry destroying such genericity comes from the result given in proposition 6, assuring $\tilde{f}_1(\bar{k}_1)\neq 0$ for small periodic potential.

Now it remains to prove that

$$d_n=\langle f_n\rangle=\frac{a}{2\pi}\int_{\mathscr{B}}f_n(k)\,\mathrm{d}k. \tag{9}$$

Following Nenciu and Nenciu (1981) we redefine the band functions up to any order of F such that we obtain a diagonal operator $\tilde{K}(F)=\mathscr{E}(F)+iF\boldsymbol{D}+O(F^\infty)$ starting from $\tilde{H}(F)=\boldsymbol{E}+F\boldsymbol{X}+iF\boldsymbol{D}$, where $\mathscr{E}(F)\sim\sum_{l=0}^{\infty}\mathscr{E}^{(l)}F^l$. We look for a unitary family of matrices $U(k)$ such that formally

$$U^{-1}(\boldsymbol{E}+F\boldsymbol{X}+iF\boldsymbol{D})U=\mathscr{E}+iF\boldsymbol{D} \tag{10}$$

that is

$$U^{-1}(\boldsymbol{E}+F\boldsymbol{X})U+FU^{-1}(i\boldsymbol{D}U)=\mathscr{E}. \tag{11}$$

† Series (7) is uniformly convergent, for n fixed, since the nth gap is open and $X_{n,m}(k):=(\tilde{W}_n(k))_{n,m}$ for each m, where the matrix $\tilde{W}_n$ is bounded.

We introduce a formal iteration method of solution which will be extensively discussed in a further paper. Let $U=\Pi_{r=1}^{\infty} U_r$, where

$$U_r^{-1}U_r=\mathbb{1} \qquad \text{and} \qquad U_r \sim \mathbb{1}+\sum_{s=r}^{\infty} F^s U_r^{(s)} \qquad \text{as } F\to 0 \tag{12}$$

so that we have $\boldsymbol{D}U_r=F^r\boldsymbol{D}U_r^{(r)}+O(F^{r+1})$. Moreover each U_r is to be determined so that

$$U_1^{-1}(\boldsymbol{E}+F\boldsymbol{X})U_1=\mathscr{E}_1 \qquad U_r^{-1}(\mathscr{E}_{r-1}+F\boldsymbol{X}_{r-1})U_r=\mathscr{E}_r \tag{13}$$

where $\mathscr{E}_r$ is a diagonal matrix for each r and $\boldsymbol{X}_r:=U_r^{-1}(\mathrm{i}\boldsymbol{D}U_r)$.

As a consequence, it is easy to obtain at least the following asymptotic estimates

$$\boldsymbol{X}_r=\mathrm{i}F^r\boldsymbol{D}U_r^{(r)}+O(F_{r+1}) \qquad \text{and} \qquad \mathscr{E}_r-\mathscr{E}_{r-1}=O(F^r) \tag{14}$$

which make consistent the assumption (12).

For our purposes it is enough to note that $U_1^{(1)}=\mathrm{i}F\Gamma(\boldsymbol{X})$, where $(\Gamma(\boldsymbol{X}))_{n,m}:=\mathrm{i}X_{n,m}/(E_n-E_m)$ is the Friedrichs operator of $\boldsymbol{X}$ with respect to $\boldsymbol{E}$. Hence, denoting by A^{D} the diagonal part of A, i.e. $(A^{\mathrm{D}})_{n,m}:=A_{n,m}\delta_n^m$, we have

$$\begin{aligned}\mathscr{E}&=\mathscr{E}_1+F\boldsymbol{X}_1^{\mathrm{D}}+O(F^3)=\mathscr{E}_1+F^2(\mathrm{i}\boldsymbol{D}U_1^{(1)})^{\mathrm{D}}+O(F^3)\\&=\mathscr{E}_1-F^2\Gamma(\boldsymbol{X})^{\mathrm{D}}+O(F^3)=\mathscr{E}_1+O(F^3)\end{aligned} \tag{15}$$

since $\Gamma(\boldsymbol{X})^{\mathrm{D}}=0$. It turns out that, up to second order, it is enough to compute $\mathscr{E}_1$, that is to diagonalize $\boldsymbol{E}+F\boldsymbol{X}$. The perturbative theory gives us

$$\mathscr{E}=\mathscr{E}_1+O(F^3)=\boldsymbol{E}+F\boldsymbol{X}^{\mathrm{D}}+\mathscr{E}_1^{(2)}F^2+O(F^3)=\boldsymbol{E}+\mathscr{E}_1^{(2)}F^2+O(F^3) \tag{16}$$

where

$$(\mathscr{E}_1^{(2)})_{n,n}=(\mathscr{E}_1^{(2)}(k))_{n,n}=\sum_{m\neq n}\frac{|X_{m,n}(k)|^2}{E_n(k)-E_m(k)}. \tag{17}$$

Following Wannier, the ladder is given by

$$\begin{aligned}\lambda_{n,j}(F)&=\langle(\mathscr{E})_{n,n}\rangle+jaF\\&=\langle E_n\rangle+jaF+\langle(\mathscr{E}_1^{(2)})_{n,n}\rangle F^2+O(F^3)\\&=\langle E_n\rangle+jaF+\left\langle\sum_{m\neq n}\frac{|X_{m,n}|^2}{E_n-E_m}\right\rangle F^2+O(F^3)\end{aligned} \tag{18}$$

whence (9) follows, and $c_n\neq d_n$. □

Now, we compare the asymptotic behaviour of $\lambda_{n,j}(F+\mathrm{i}\eta)$ as $F+\mathrm{i}\eta\to 0$ at each complex direction and the one at the real direction as $F\to 0^+$. In particular, admitting to extend analyticity of $\lambda_{n,j}(F+\mathrm{i}\eta)$ from the disk $B_{R_n}(\mathrm{i}R_n)$ up to the real direction, for $F>0$ small, with continuity on the boundary, the inequality $c_n\neq d_n$ implies the existence of a strong divergence sequence for $\lambda_{n,j}$ tangent to the real axis at the origin. This follows from classical results as the Phragmén-Lindelöf theorem.

Let us stress that the existence of such strong divergence sequence seems to be less natural than the existence of a horn of singularities of $\lambda_{n,j}(F+\mathrm{i}\eta)$.

Proposition 2. Let $S_{\varepsilon,\delta} := \{z \in \mathbb{C} | 0 < \arg z < \varepsilon, |z| < \delta\}$, $\varepsilon > 0$, $\delta > 0$. If the eigenvalue $\lambda_{n,j}(z)$ is analytic in $S_{\bar{\varepsilon},\bar{\delta}}$ and continuous in $\overline{S_{\bar{\varepsilon},\bar{\delta}}} - \{0\}$, for some $\bar{\varepsilon} > 0$ and $\bar{\delta} > 0$, then there exists a sequence $\{z_l\}_{l \in \mathbb{N}}$, $z_l \in S_{\bar{\varepsilon},\bar{\delta}}$, $\lim_{l\to+\infty} z_l = 0$ and $\lim_{l\to+\infty} \arg(z_l) = 0$ such that, for any $p > 0$,

$$\lim_{l\to+\infty} |\lambda_{n,j}(z_l)| \exp[-|z_l|^{-p}] = +\infty. \tag{19}$$

Proof. Let $\varepsilon = \pi/p < \bar{\varepsilon}$. By proposition 1 and by the hypothesis, the function

$$\Lambda_n(z) := z^{-2}[\lambda_{n,j}(z) - \langle E_n \rangle - jaz] \tag{20}$$

is analytic in $S_{\varepsilon,\bar{\delta}}$ and continuous in $\overline{S_{\varepsilon,\bar{\delta}}} - \{0\}$, moreover has different real limits on the boundaries of the sector $S_{\varepsilon,\bar{\delta}}$, in fact:

$$\lim_{\arg z = \varepsilon,\, z\to 0} \Lambda_n(z) = c_n \qquad \lim_{\arg z = 0,\, z\to 0} \Lambda_n(z) = d_n.$$

Hence, by the Phragmén-Lindelöf Theorem (see Hille 1962), there exists a sequence $v_{l,p} \in S_{\varepsilon,\delta} \subset S_{\bar{\varepsilon},\bar{\delta}}, l \in \mathbb{N}$, such that $\lim_{l\to+\infty} v_{l,p} = 0$ and $\Lambda_n(v_{l,p}) = O(\exp[|v_{l,p}|^{-p}])$ as $l \to +\infty$. So, choosing e.g. $l = p$, the sequence $z_l = v_{l,l}$ satisfies proposition 2. □

Remark 3. Actually we have no explicit example of a function satisfying all the conditions of propositions 1 and 2, so that the class of such functions could be void. In such a case the only possible conclusion is the following one: the origin is the limit point of a sequence of singularities of $\lambda_{n,j}(z)$ belonging to the domain $D_n = \{z \in \mathbb{C} | \Im z > 0, \Re z > 0, z \notin B_{R_n}(\mathrm{i}R_n)\}$ (i.e. there exists a horn of singularities for $\lambda_{n,j}(z)$).

Remark 4. If the eigenvalue $\lambda_{n,j}(z)$ is meromorphic in $\tilde{\bar{S}}_{\varepsilon,\delta} := \{z \in \mathbb{C} | -\varepsilon < \arg z < \varepsilon, |z| < \delta\}$ then the positive direction is a Julia direction (see Hille 1962, chapter 15, section 4) at 0, i.e.: $\lambda_{n,j}(z)$ on $S_{\varepsilon,\delta}$, $\forall \varepsilon, \delta > 0$, omits at most two values of $\bar{\mathbb{C}}$. This means that in the hypothesis of analyticity on the real axis, we have a wild behaviour of the function in a neighbourhood of the real axis. In such a case we have an example partially fulfilling the hypothesis: the function $f(z) = \tan(z^{-1})$ exhibiting a sequence of singularities on the real axis.

Remark 5. As recalled in the introduction an explicit example exhibiting the same phenomenon of different asymptotic behaviours in different directions in the sectors $\arg g \in [3\pi/2, 3\pi/2 + \beta]$, $0 < \beta < 3\pi$, is given by the eigenvalues of the arharmonic oscillator $T(g) = p^2 - x^2 + gx^4$. In such a case the existence of a horn of singularities given by the crossings of levels (see Bender and Wu 1969, Simon 1970, Crutchfield 1978) is completely proved.

Finally, we give the behaviour, when the periodic potential tends to zero, of the second order coefficients c_1 and d_1 of the asymptotic expansions (4) and (5) in the complex and real electric field respectively, where

$$c_1 = f_1(\bar{k}_1) \qquad d_1 = \langle f_1 \rangle,$$

$\bar{k}_1$ is defined by $E_1(\bar{k}_1) = \langle E_1 \rangle$, $0 < \bar{k}_1 < \pi/a$ and $f_1(k)$ is defined in (7). Such a result agrees with and extends the statement of proposition 1.

Let us set $V(x)=\beta V_1(x)$, $0<\beta\leqslant 1$. The following proposition holds:

Proposition 6. If the periodic potential is small enough then $c_1\neq d_1$. In particular:

$$\lim_{\beta\downarrow 0} c_1=0 \qquad \text{and} \qquad \lim_{\beta\downarrow 0} d_1=-\infty. \tag{21}$$

Proof. Following Wannier let us consider the Schrödinger equation for the Fourier coefficients of the Bloch waves

$$(K+k)^2 w_n^k(K)+\sum_{j\in\mathbb{Z}} \tilde{V}_{K-J} w_n^k(J)=E_n(k)w_n^k(K) \qquad k\in\mathcal{B},\ K\in\mathbb{Z}. \tag{22}$$

Performing the derivative in (22) with respect to the quasi-momentum k we obtain the well known expression

$$X_{n,m}(k)=-2\mathrm{i}\frac{\langle w_n^k, \boldsymbol{K} w_m^k\rangle_{l^2}}{E_n(k)-E_m(k)}. \tag{23}$$

From a second derivative of (22) we obtain

$$\frac{\mathrm{d}^2E_n}{\mathrm{d}k^2}\equiv E_n''(k)=2+2\sum_{m\neq n}|X_{n,m}(k)|^2[E_n(k)-E_m(k)]. \tag{24}$$

For $k\in(0,\pi/a)$, as $\beta\downarrow 0$, we have that $E_n(k)$ tends to $[(-1)^{n-1}k+2\nu\pi/a]^2$, here $\nu:=[n/2]$ is the integer part of $n/2$, $E_n'(k)$ tends to $2(-1)^{n-1}[(-1)^{n-1}k+2\nu\pi/a]$, $E_n''(k)$ tends to 2 and so

$$\lim_{\beta\downarrow 0}\sum_{m\neq n}|X_{n,m}(k)|^2[E_n(k)-E_m(k)]=0 \qquad \forall k\in(0,\pi/a). \tag{25}$$

Now we are ready to prove the statements in proposition 6.

From (7) we have that

$$\begin{aligned} 0\geqslant f_1(k)&=\sum_{m\neq 1}\frac{|X_{1,m}(k)|^2}{E_1(k)-E_m(k)}\\ &\geqslant\frac{1}{[E_1(k)-E_2(k)]^2}\sum_{m\neq 1}|X_{1,m}(k)|^2[E_1(k)-E_m(k)]\to 0 \qquad \text{as } \beta\downarrow 0 \end{aligned} \tag{26}$$

for any $k\in(0,\pi/a)$, in particular we have that $c_1:=f_1(\bar{k}_1)\to 0$ as $\beta\downarrow 0$ being $\bar{k}_1\in(0,\pi/a)$.

In contrast, the term $d_1:=\langle f_1\rangle$ is arbitrarily large when the periodic potential tends to zero. In fact, for any $h>0$ we have that

$$\langle f_1\rangle=\frac{a}{\pi}\int_0^{\pi/a} f_1(k)\,\mathrm{d}k\leqslant\frac{a}{\pi}\int_{\pi/a-h}^{\pi/a} f_1(k)\,\mathrm{d}k.$$

In order to estimate the latter integral we give the useful estimate obtained from (22) and (23):

$$\begin{aligned} \sum_{m=3}^{\infty}&|X_{1,m}(k)|^2[E_m(k)-E_1(k)]\\ &\leqslant\frac{\sum_{m=2}^{\infty}|X_{1,m}(k)|^2[E_1(k)-E_m(k)]^2}{E_3(k)-E_1(k)}\leqslant\frac{4\sum_{m=1}^{\infty}|\langle w_1^k,\boldsymbol{K}w_m^k\rangle_{l^2}|^2}{E_3(k)-E_1(k)}\\ &=\frac{4|\langle w_1^k,\boldsymbol{K}w_1^k\rangle_{l^2}|^2}{E_3(k)-E_1(k)}\leqslant\frac{4E_1(k)+O(\beta)}{E_3(k)-E_1(k)}\leqslant c_1 \end{aligned} \tag{27}$$

where c_1 denotes a constant. Hence

$$\left|\frac{a}{\pi}\int_{\pi/a-h}^{\pi/a} f_1(k)\,\mathrm{d}k\right|$$

$$\geqslant \frac{a/\pi\int_{\pi/a-h}^{\pi/a}|X_{1,2}(k)|^2[E_2(k)-E_1(k)]\,\mathrm{d}k}{[E_1(\pi/a-h)-E_2(\pi/a-h)]^2}$$

$$\geqslant \frac{a/2\pi\int_{\pi/a-h}^{\pi/a}[2-E_1''(k)-c_1]\,\mathrm{d}k}{[E_1(\pi/a-h)-E_2(\pi/a-h)]^2}$$

$$=\frac{a}{2\pi}\,\frac{[-E_1'(\pi/a)+E_1'(\pi/a-h)+(2-c_1)h]}{[E_1(\pi/a-h)-E_2(\pi/a-h)]^2}$$

$$\to \frac{a^2-a^3c_1h/2\pi}{16\pi^2h^2} \qquad \text{as } \beta\downarrow 0 \tag{28}$$

since $E_1'(\pi/a)=0$ and $E_n(k)\to[(-1)^{n-1}k+2\nu\pi/a]^2$, $\nu=[n/2]$, as $\beta\downarrow 0$ and $k\in(0,\pi/a)$. Proposition 6 is completely proved since $h>0$ is arbitrary. □

Conclusion

The existence of different asymptotics in the real with respect to the complex directions gives evidence for a horn of singularities for Stark–Wannier ladders similar to the Bender–Wu ones already observed in the finite crystal case. We have left, as a possible but improbable case, the alternative of the existence of a sequence of strong divergence for such eigenvalues. In any case the effect of the behaviour of the width of each resonance, not determined by the asymptotic expansion, should not be less dramatic.

Acknowledgment

It is a pleasure to thank Professor François Bentosela for many discussions on such problems.

References

Avron J 1979 On the spectrum of $p^2+V(x)+\epsilon x$, with V periodic and ϵ complex *J. Phys. A: Math. Gen.* **12** 2393

Avron J 1982 The lifetime of Wannier ladder states *Ann. Phys.* **143** 33

Bender C M and Wu T T 1969 Anharmonic oscillator *Phys. Rev.* **184** 1231

Bentosela F, Caliceti E, Grecchi V, Maioli M and Sacchetti A 1988 Analyticity and asymptotics for the Stark–Wannier states *J. Phys. A: Math. Gen.* **21** 3321

Bentosela F and Grecchi V 1990 Stark–Wannier ladders *Commun. Math. Phys.* in press

Bentosela F, Grecchi V and Zironi F 1982a Approximate ladder of resonances in a semi-infinite crystal *J. Phys. C: Solid State Phys.* **15** 7119

—— 1982b Oscillations of Wannier resonances *Phys. Rev. Lett.* **50** 84

Berezhkovskii A M and Ovchinnikov A A 1976 True width of electron levels in a crystal in a static electric field *Sov. Phys. Solid State* **18** 1908

Bessis D, Marmi S and Turchetti G 1989 On the singularities of divergent series arising from normal form theory *Rendiconti di Matematica (Roma)* VII-9 p 645

Buslaev V S and Dmitrieva L A 1987 Adiabatic perturbation of periodic potential II *Teor. Mat. Fiz.* **73** 430

Buslaev V S and Dmitrieva L A 1989 Bloch electrons in an external electric field *Schrödinger Operator: Standard and Non-standard* ed P Exner and P Seba (Singapore: World Scientific)

Combes J M and Hislop P D 1990 Quantum tunnelling, non-trapping and the existence of Stark ladder resonances *J. Phys. A: Math. Gen.* **23** 1501

Crutchfield II W Y 1978 No horn of singularities for double-well anharmonic oscillators *Phys. Lett.* **71B** 109

Ferrari M, Grecchi V and Zironi F 1985 Stark–Wannier states and Bender–Wu singularities *J. Phys. C: Solid State Phys.* **18** 5825

Herbst I and Howland J 1981 The Stark ladder and other one-dimensional external electric field problems *Commun. Math. Phys.* **80** 23

Hille E 1962 *Analytic function theory* vol II (Waltham, MA: Blaisdell Publishing Company)

Nenciu A and Nenciu G 1981 Dynamics of Bloch electrons in external electric field *J. Phys. A: Math. Gen.* **14** 2817

Simon B 1970 Coupling constant analyticity for the anharmonic oscillator *Ann. Phys.* **58** 76

Wannier G H 1960 Wave functions and effective Hamiltonian for Bloch electrons in an electric field *Phys. Rev.* **117** 432